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Quantum Computing: Recap

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Contents

- Quantum Mechanics Refresher
  - states, operators, evolution, measurements
  - composite quantum systems, direct product structure, entanglement

- Quantum Algorithms for perfect Quantum Computers
  - Deutsch’s problem and Simon’s problem
  - quantum circuits (teleportation)
  - quantum computers can do any classical computation
  - simulating quantum computers is exceedingly hard
  - Quantum Fourier transform
  - phase estimation

Contents

- Quantum Algorithms for perfect Quantum Computers
  - Quantum Fourier transform, phase estimation
  - Shor’s algorithm
  - HHL algorithm (solving systems of linear equations)

- Quantum Error Correction
  - repetition codes stabilizer formalism, surface codes

- NISQ Quantum Computing
  - variational algorithms
  - QAOA
  - Cirq

Composite Quantum Systems

superposition of states is a possible state
for qubits:

2 $$|\psi\rangle = c_{00}\,|00\rangle + c_{01}\,|01\rangle + c_{10}\,|10\rangle + c_{11}\,|11\rangle$$

3 $$\begin{aligned}|\psi\rangle &= c_{000}\,|000\rangle + c_{001}\,|001\rangle + c_{010}\,|010\rangle + c_{011}\,|011\rangle \\ &= c_{100}\,|100\rangle + c_{101}\,|101\rangle + c_{110}\,|110\rangle + c_{111}\,|111\rangle\end{aligned}$$

4 $$\begin{aligned}|\psi\rangle &= c_{0000}\,|0000\rangle + c_{0001}\,|0001\rangle + c_{0010}\,|0010\rangle + c_{0011}\,|0011\rangle \\ &= c_{0100}\,|0100\rangle + c_{0101}\,|0101\rangle + c_{0110}\,|0110\rangle + c_{0111}\,|0111\rangle \\ &= c_{1000}\,|1000\rangle + c_{1001}\,|1001\rangle + c_{1010}\,|1010\rangle + c_{1011}\,|1011\rangle \\ &= c_{1100}\,|1100\rangle + c_{1101}\,|1101\rangle + c_{1110}\,|1110\rangle + c_{1111}\,|1111\rangle\end{aligned}$$

Composite Quantum Systems

entanglement is important

$$|\psi\rangle = (a_0|0\rangle + a_1|1\rangle) \otimes (b_0|0\rangle + b_1|1\rangle)$$
$$= a_0b_0|00\rangle + a_0b_1|01\rangle + a_1b_0|10\rangle + a_1b_1|11\rangle$$

$$|\psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle$$

could express exponential amount of number with much less numbers

reduced density matrix of (maximally) entangled state

$$\langle 0| \, |\phi\rangle\langle\phi| \, |0\rangle + \langle 1| \, |\phi\rangle\langle\phi| \, |1\rangle = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$$

no local information

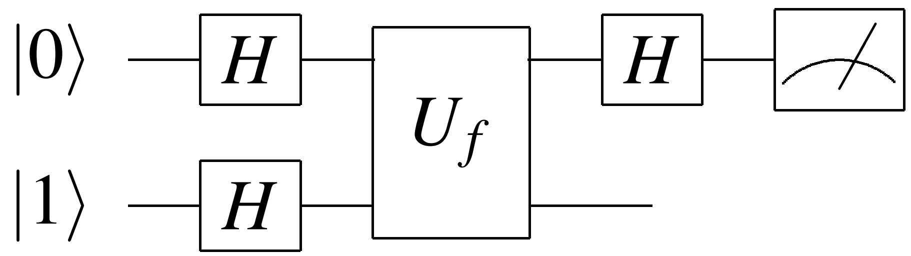
Quantum Algorithms

in this part assumed:

- gates work perfectly
- qubits stay in the same state if not acted upon

  →infinite coherence times

example:



often initially
Hadamards on all qubits

$$H^n |x\rangle = \frac{1}{2^{n/2}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle$$

$$x = x_0, x_1, \ldots, x_n$$

superposition of all basis states

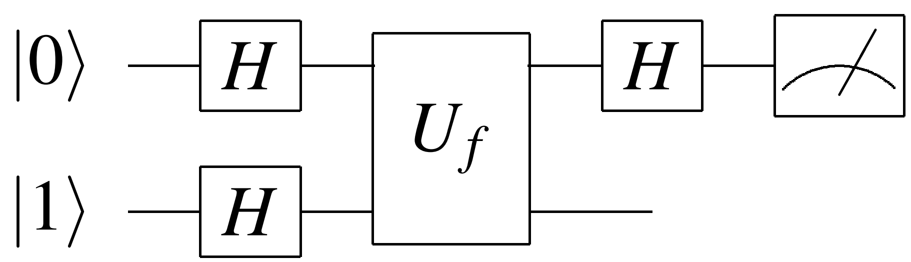
Quantum Algorithms

in this part assumed:

- gates work perfectly
- qubits stay in the same state if not acted upon

$\rightarrow$infinite coherence times

example:



make sure no correlations between qubits before readout

for maximally entangled state:

$$\langle 0|\ |\phi\rangle\langle\phi|\ |0\rangle + \langle 1|\ |\phi\rangle\langle\phi|\ |1\rangle = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$$

measurement (readout) produces random result

Quantum Algorithms

quantum evolution unitary $\rightarrow$ algorithms reversible

$$|\psi_f\rangle = U\,|\Psi_i\rangle$$

$\rightarrow$ decompose U into single- and two-qubit gates

$\rightarrow$ is the only stuff that can be implemented

$\rightarrow$ entangling two-qubit gates are important, e.g. CNOT

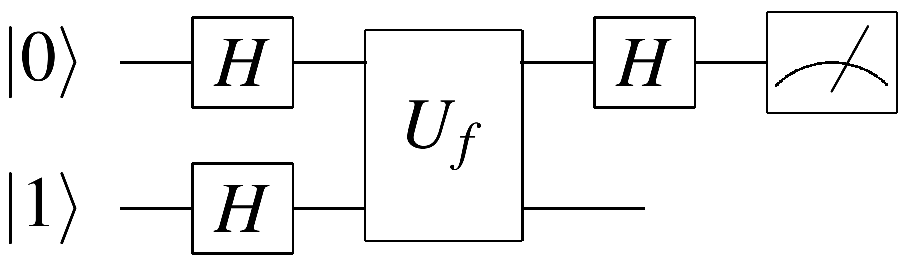
$$(\alpha\,|0\rangle + \beta\,|1\rangle) \otimes |0\rangle \;\rightarrow\; \alpha\,|00\rangle + \beta\,|11\rangle$$

arbitrary single-qubit gates plus CNOT is universal

$\rightarrow$ can do any computation

Deutsch's and Simon's Problems

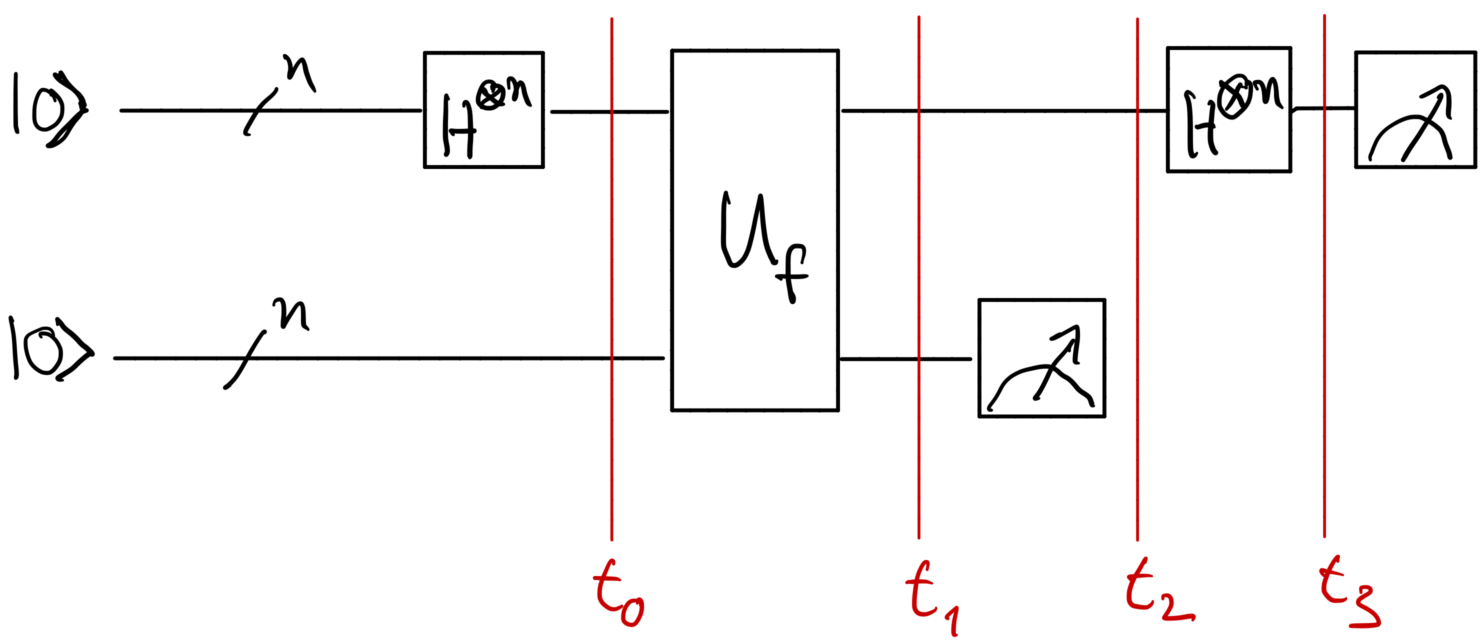
Deutsch: $f(0) = f(1)$?



$$U_f\,|j\rangle\,|k\rangle = |j\rangle\,|f(j)\oplus k\rangle$$

class.: 2 queries
quantum: 1 query

Simon: find period s of function ($f(x) = f(y)$ for $y = x \oplus s$)

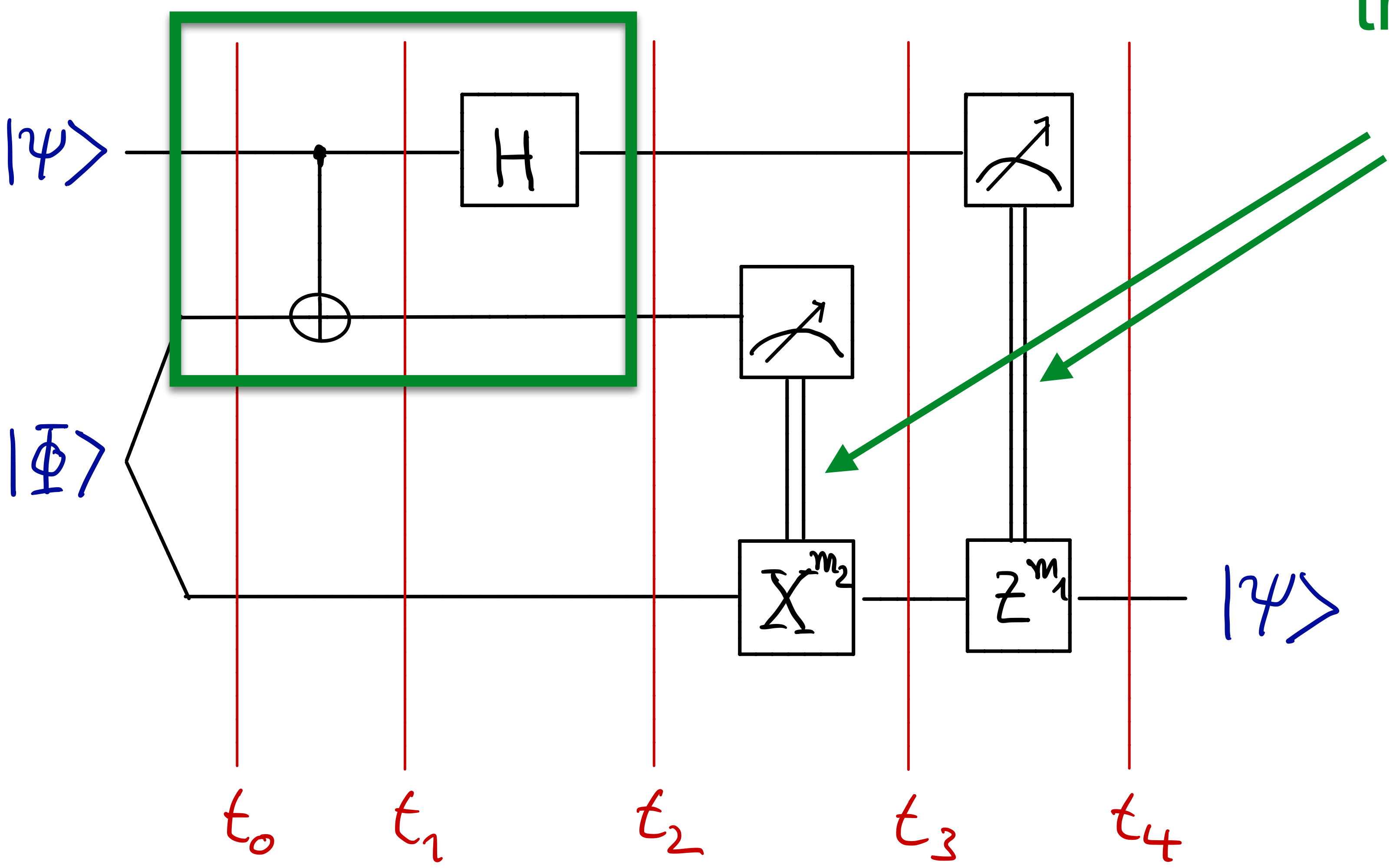


$$U_f\,|j\rangle\,|k\rangle = |j\rangle\,|f(j)\oplus k\rangle$$

class.: $2^{n/2}$ queries
quantum: $\Omega(n)$ queries

Quantum Teleportation

state transfer circuit



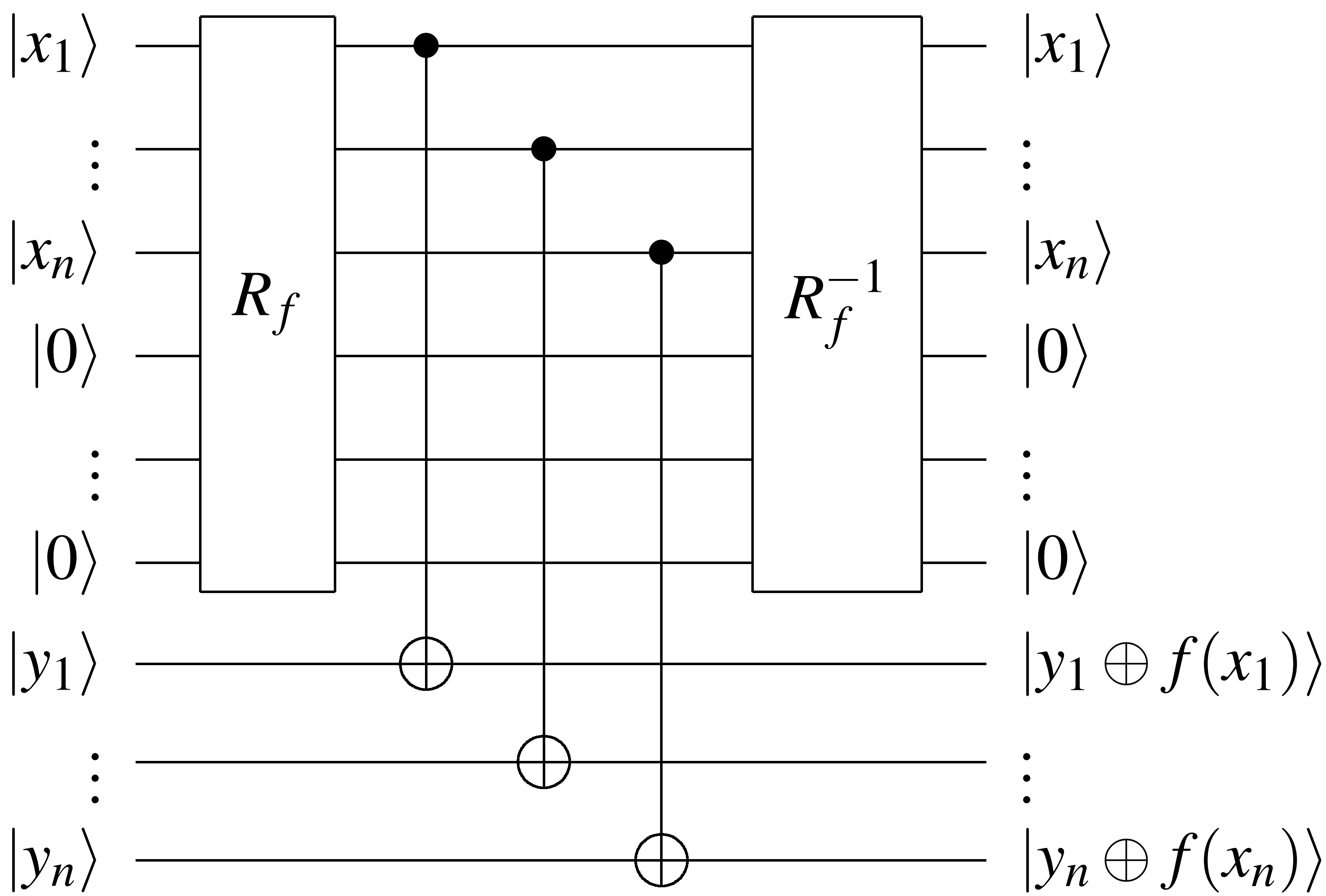
transfer does not happen before classical information is transmitted!

Classical vs Quantum

any classical algorithm can be implemented on a quantum computer

undo correlations

extra qubits
→reversible,
replace AND
with Tofoli



copy results to extra qubits

Classical vs Quantum

simulating a quantum computer requires exponential resources

Schrödinger

$$|\psi\rangle = \sum_{j_1, j_2, \ldots j_n = 0}^{1} \alpha_{j_1, j_2, \ldots j_n} |j_1, j_2, \ldots j_n\rangle = \sum_{\vec{j}} \alpha_{\vec{j}} |\vec{j}\rangle$$

$$\frac{d}{dt}\alpha_{\vec{j}} = \sum_{\vec{l}} H_{\vec{j};\vec{l}}\, \alpha_{\vec{l}}$$

exponential amount of RAM

Feynman

$$\langle \vec{j}_f | U | \vec{j}_i \rangle = \langle \vec{j}_f | U_L U_{L-1} \ldots U_2 U_1 | \vec{j}_i \rangle$$

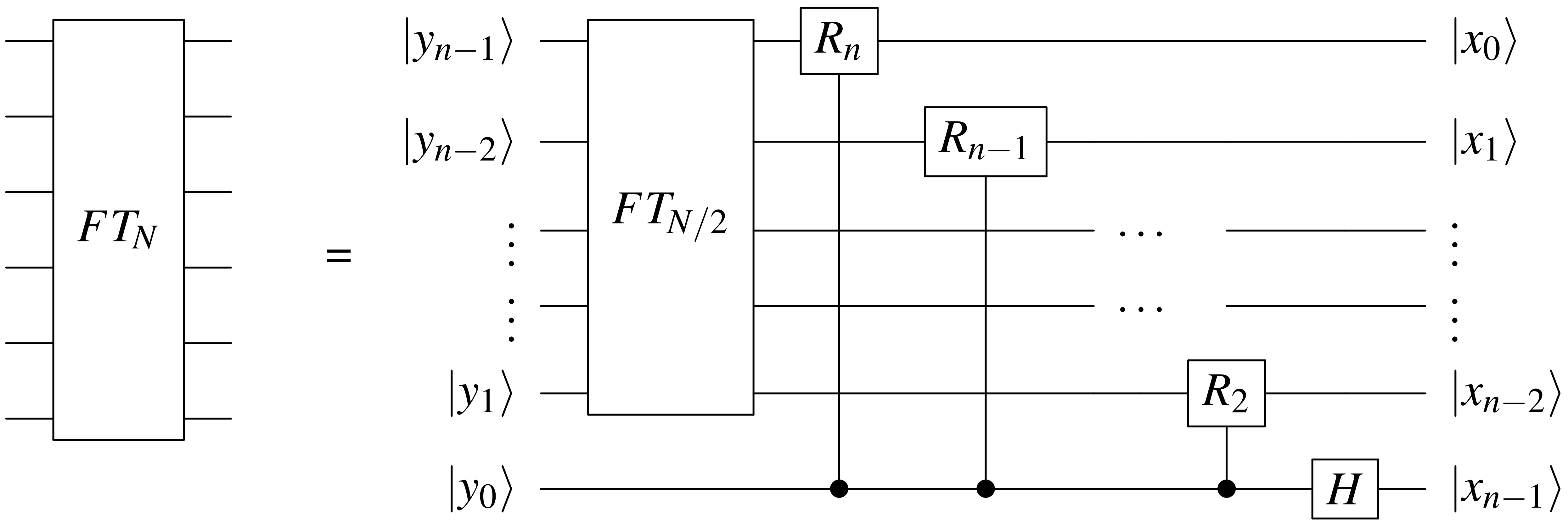
$$= \sum_{\vec{j}_{L-1}} \sum_{\vec{j}_{L-2}} \cdots \sum_{\vec{j}_1} \langle \vec{j}_f | U_L | \vec{j}_{L-1} \rangle \, \langle \vec{j}_{L-1} | U_{L-1} | \vec{j}_{L-2} \rangle \ldots \langle \vec{j}_1 | U_1 | \vec{j}_i \rangle$$

exponential amount of time

Quantum Fourier Transform

$$|\tilde{y}\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} \omega^{xy} |x\rangle \qquad \omega = e^{2\pi i/N} \qquad \omega^N = 1 \qquad \sum_{j=0}^{N-1} \omega^j = 0$$

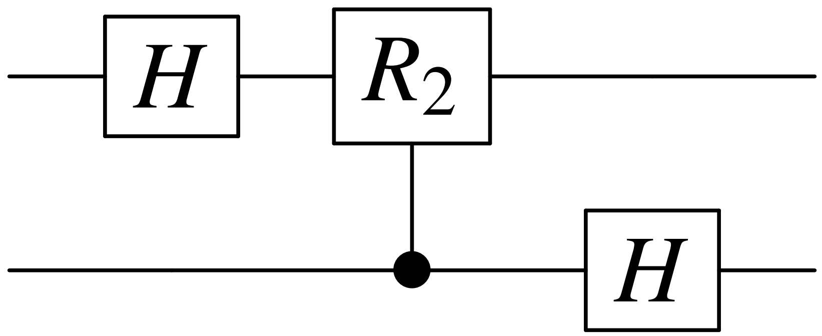
$$|x\rangle = |x_{n-1}, x_{n-2}, \ldots, x_0\rangle \qquad x = x_{n-1}2^{n-1} + x_{n-2}2^{n-2} + \cdots + x_1 2 + x_0$$



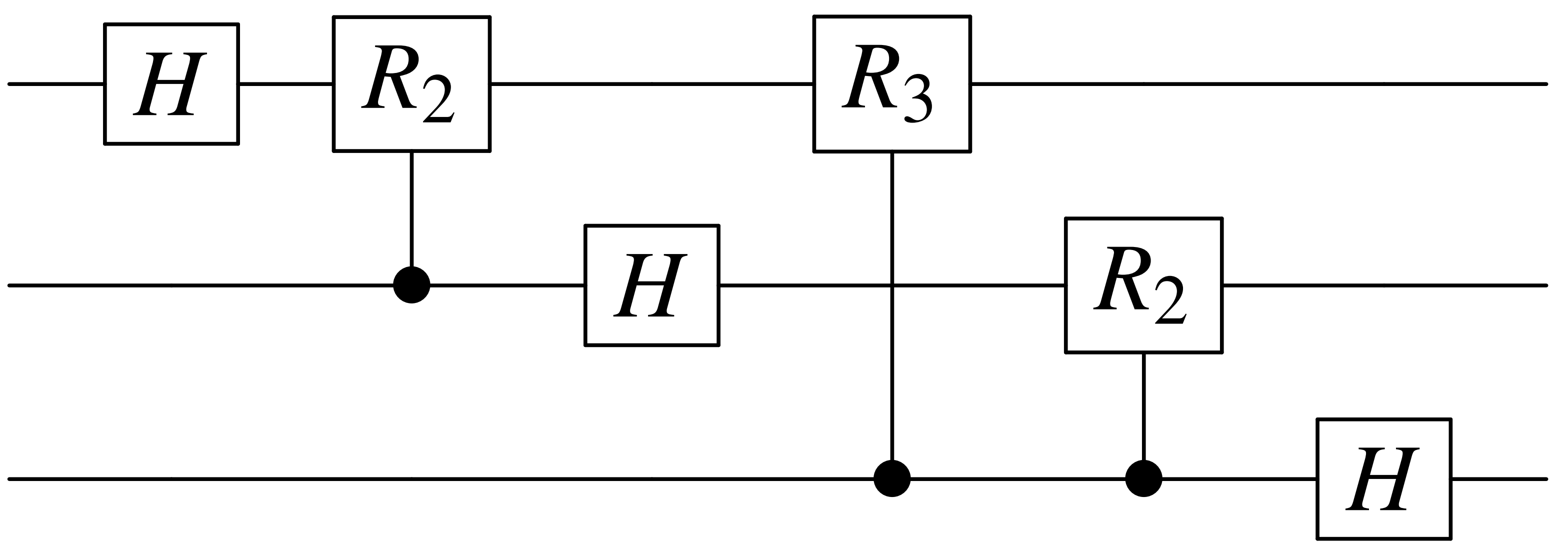
$$R_j = |0_j\rangle\langle 0_j| + e^{2\pi i 2^{-j}} |1_j\rangle\langle 1_j|$$

Quantum Fourier Transform

n=2:



n=3:



Quantum Phase Estimation

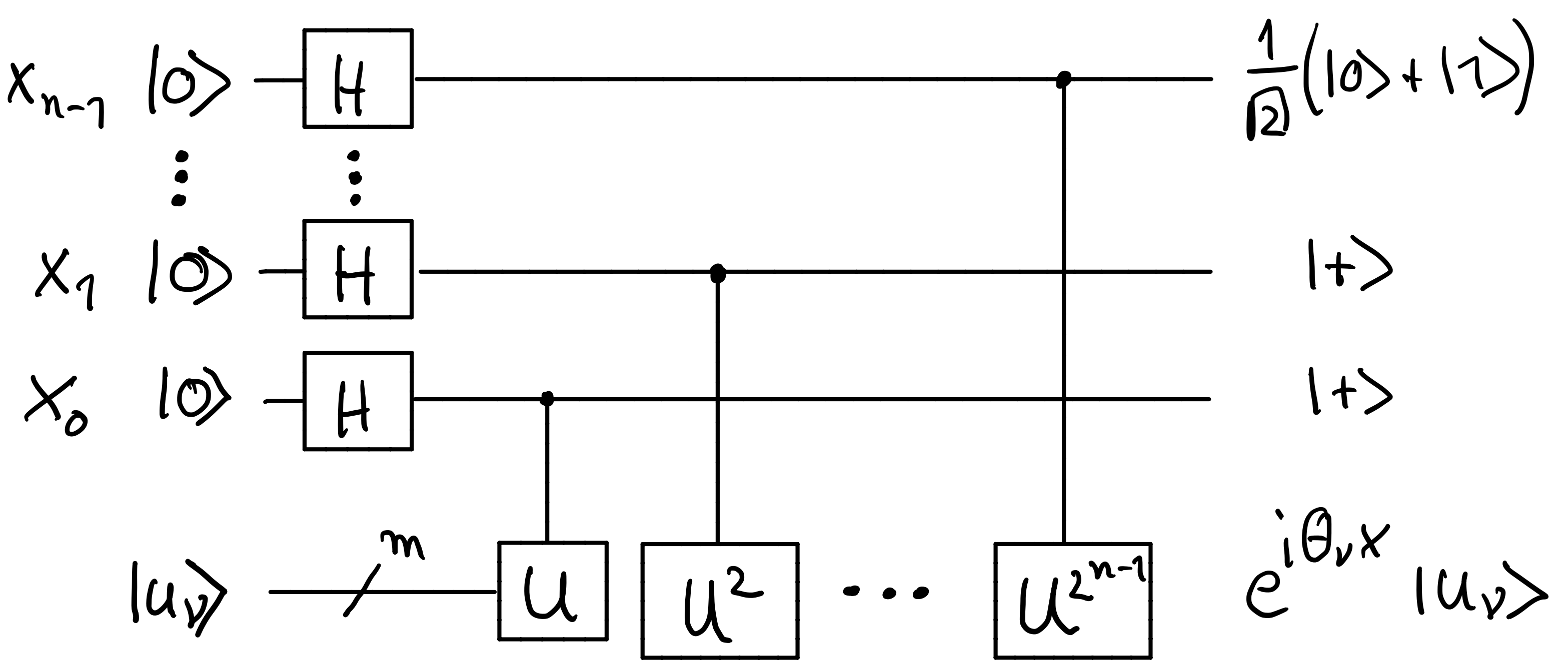
$$U\,|u_\nu\rangle = e^{i\theta_\nu}\,|u_\nu\rangle$$

find θ_ν

$$|x\rangle = |x_{n-1}, x_{n-2}, \ldots, x_0\rangle \qquad x = x_{n-1}2^{n-1} + x_{n-2}2^{n-2} + \cdots + x_1 2 + x_0$$

$$FT_N^\dagger \frac{1}{2^{n/2}} \sum_{x=0}^{N-1} e^{i\theta_\nu x}\,|x\rangle = \sum_y \left(\frac{1}{N} \sum_x \exp\left[\frac{2\pi i}{N} \left(\frac{N\theta_\nu}{2\pi} - y \right) x \right] \right) |y\rangle$$

prepare via



strongly peaked at

$$y = N\theta_\nu/(2\pi)$$

$\rightarrow$ will measure

$$y = N\theta_\nu/(2\pi)$$

Shor's Algorithm

quantum step: period finding

find period of $f(x) = a^x \bmod N \quad (a > 1, x > 0)$

$$U|x\rangle|b \bmod N\rangle = |x\rangle|b\,a \bmod N\rangle$$

$$U^x|x\rangle|b \bmod N\rangle = |x\rangle|b\,a^x \bmod N\rangle$$

controlled multiplication as in phase estimation

$\rightarrow$ apply phase estimation to U $\qquad U|u_v\rangle = e^{i\theta_v}|u_v\rangle$

can prepare $|1\rangle = |a^0\rangle$: superposition of all eigenstates of U

multiple phase estimation measurements $\rightarrow$ period of $f(x)$

HHL Algorithm

solve: $A|x\rangle = |b\rangle \quad \rightarrow \quad |x\rangle \propto A^{-1}|b\rangle$

strategy: phase estimation for $U = \exp(2\pi i \kappa A/N)$

apply controlled rotation of form:

$$|y\rangle\,|0\rangle \rightarrow |y\rangle\left(\sqrt{1-\frac{y^2}{2}}\,|0\rangle + \frac{1}{y}\,|1\rangle\right)$$

eigenstate of U

measure and keep outcome 1

$\rightarrow$ applied $1/\lambda_a$, where $A|a\rangle = \lambda_a|a\rangle$

Error Correction

gates will never be perfect → actively correct for errors

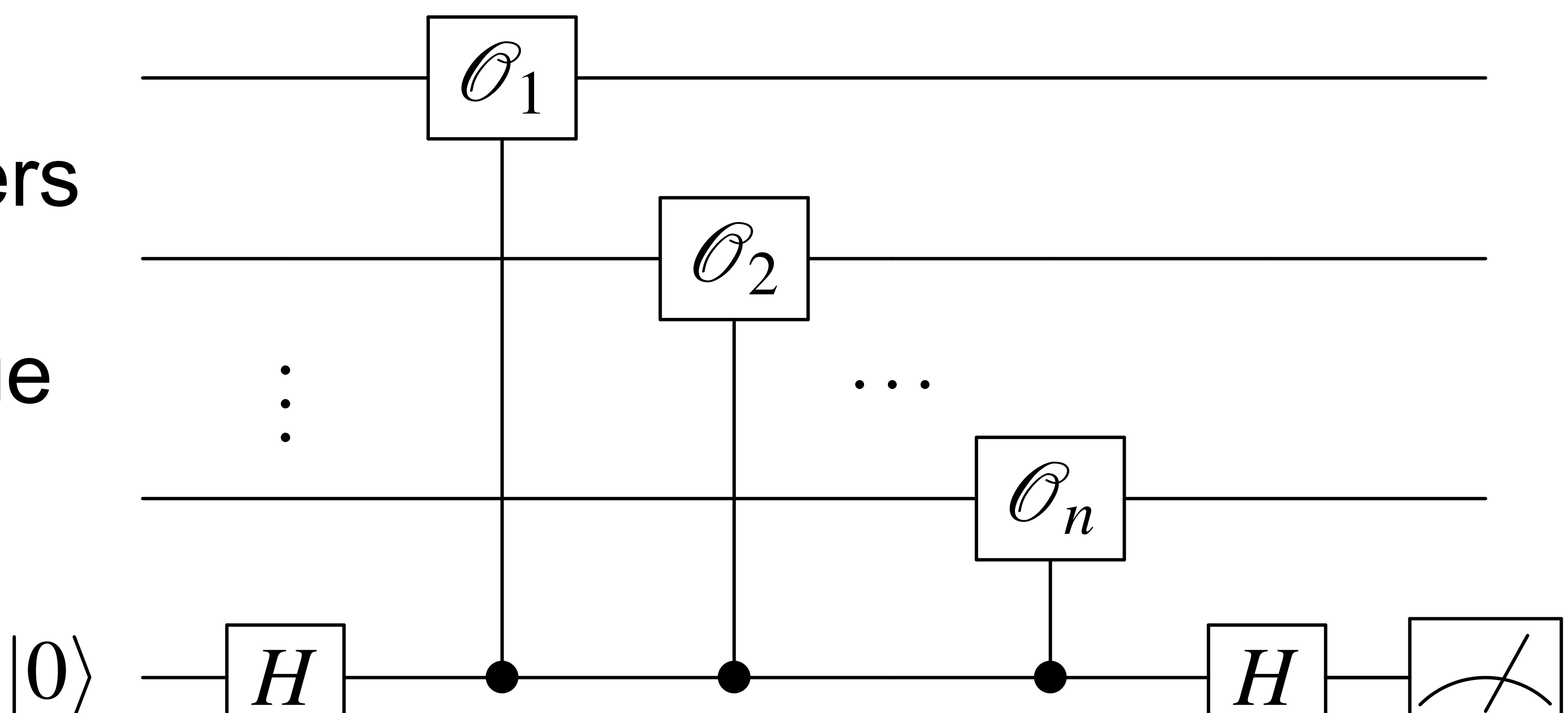
many physical qubits form a logical qubit

logical states → entangled states of many physical qubits

→ stabilizers

$$P_k \left|\psi_j\right\rangle_L = \left|\psi_j\right\rangle_L \qquad P_k = X \otimes X \otimes Y \otimes Z \otimes \mathbb{1} \otimes \ldots$$

measuring stabilizers detects errors but does not change logical states

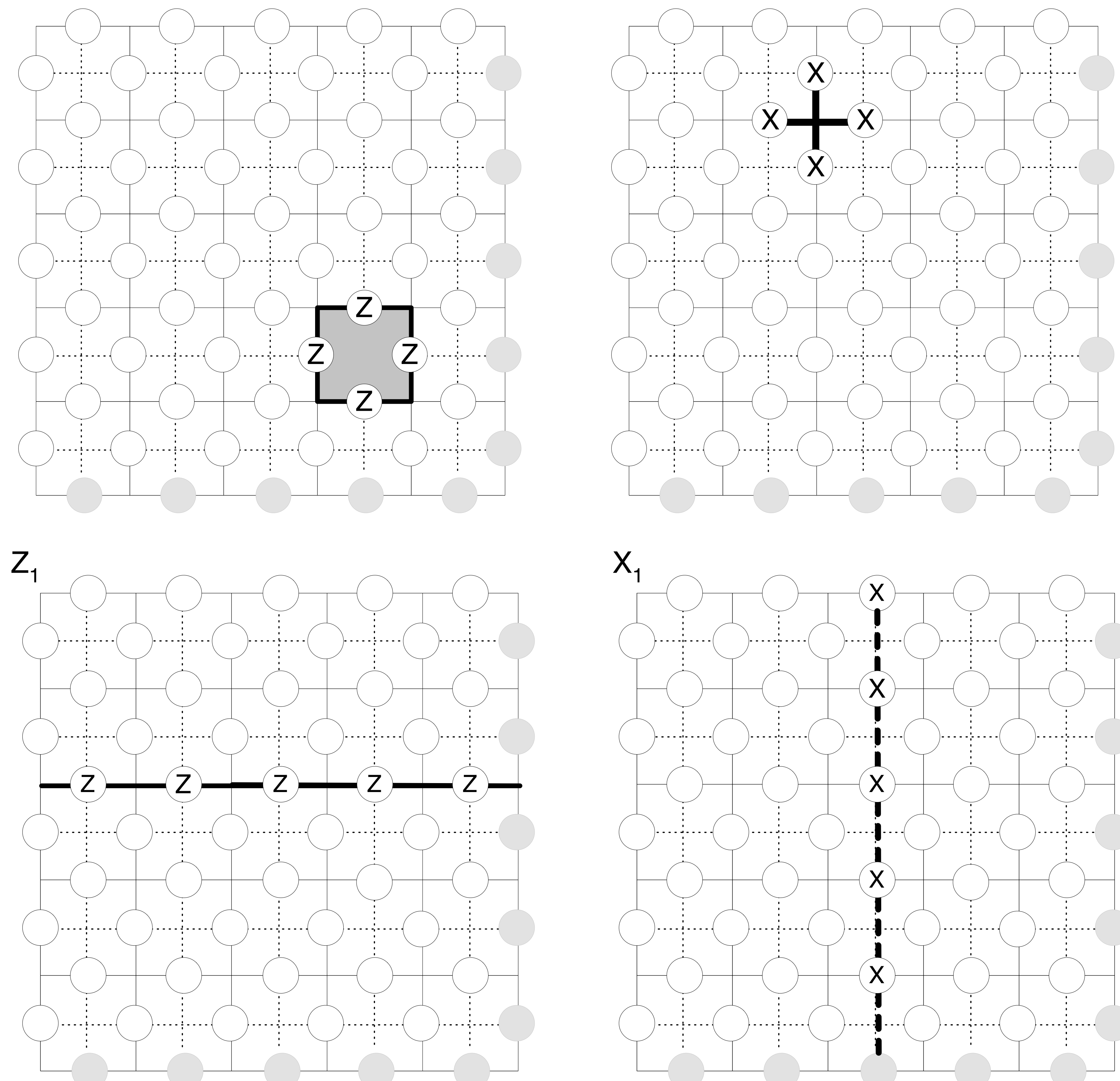


Error Correction: Toric Code

stabilizer generators

4 logical states
2 logical qubits

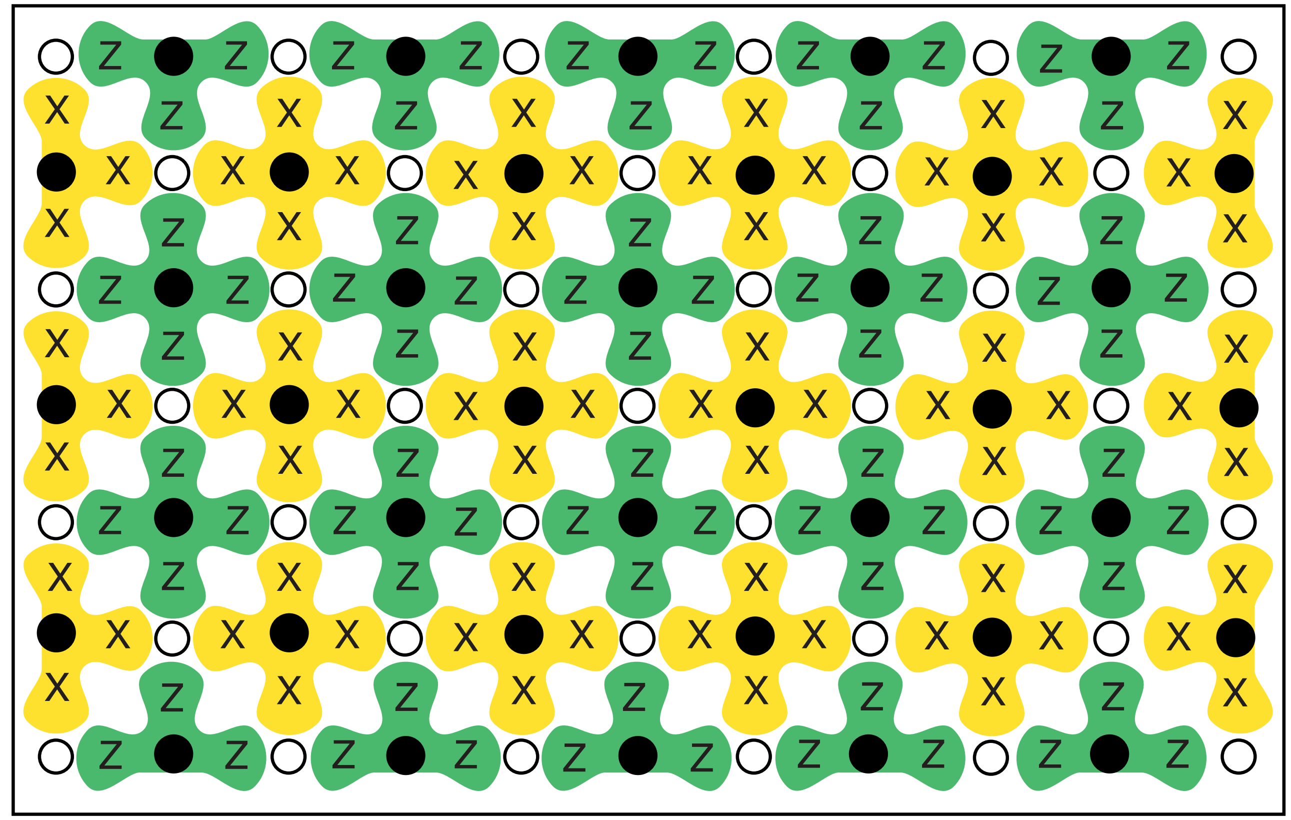
logical operators



Error Correction: Surface Code

real hardware

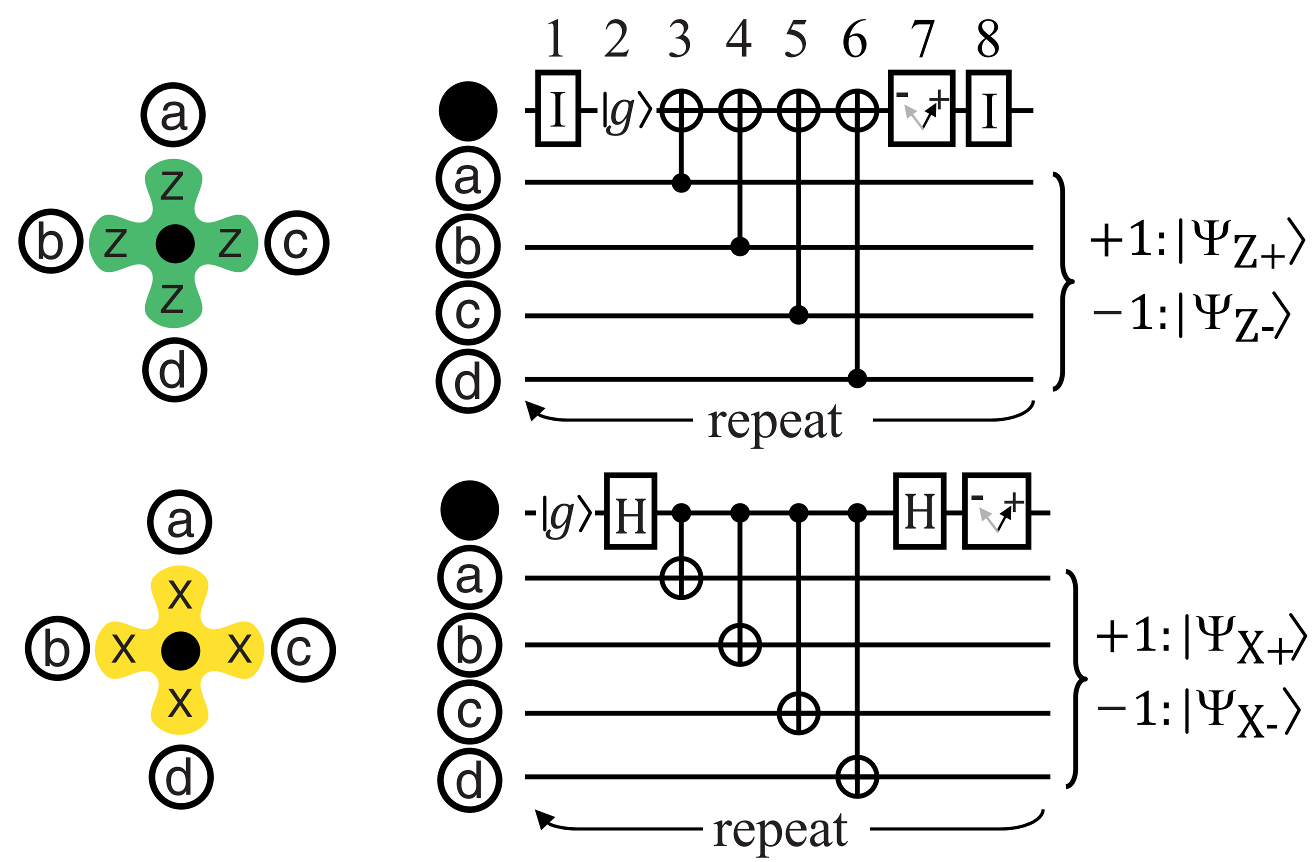
→ no periodic boundary condition



errors can also occur during error detection and correction

→ need $p_{error} < 1\,\%$

→ need 1000 physical qubits per logical qubit



Variational Algorithms

will not have error correction soon → what can we do now?

find ground state of a Hamiltonian H

- condensed matter physics
- ground or low energy state

  → stable configuration

  → quantum chemistry
- can be applied to combinatorial optimization

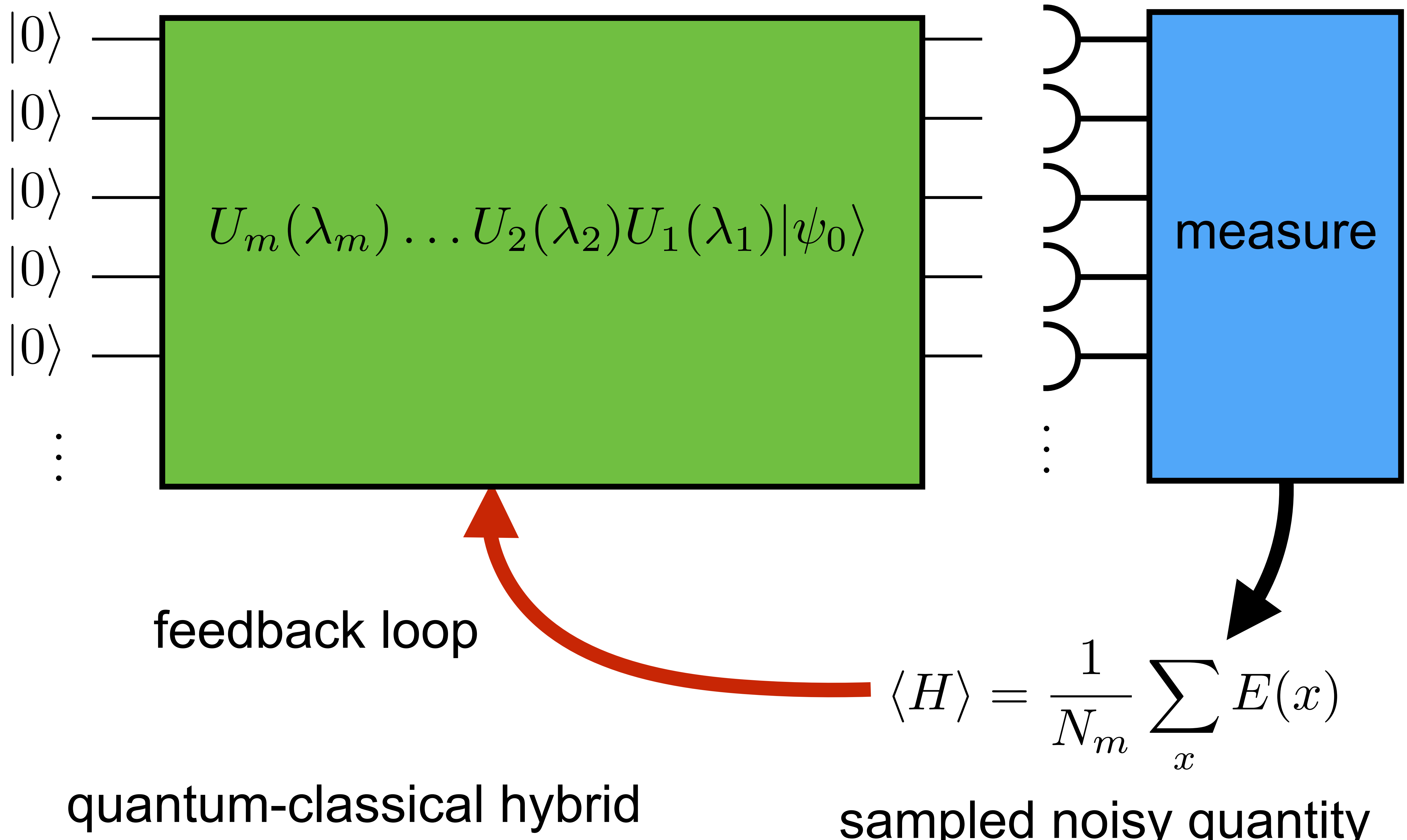
  → QAOA

$$|\psi(\vec{\theta})\rangle = U_m(\theta_m)\, U_{m-1}(\theta_{m-1}) \ldots U_1(\theta_1)\, |0, 0, \ldots, 0\rangle$$

$\theta_1, \theta_2, \ldots, \theta_m$ are variational parameters

→ minimize $E(\vec{\theta}) = \langle \psi(\vec{\theta}) | H | \psi(\vec{\theta}) \rangle$

Variational Algorithms



Quantum Approximate Optimization

combinatorial optimization problem:

find bit-string $z = z_1 z_2 \ldots z_n$ such that $C(z)$ is minimized

sum of clauses: $C(z) = -C_1(z) - C_2(z) - C_3(z) - \ldots$

all clauses true, $C_j(z) = 1$, → minimum of $C(z)$

example

$$C(z) = 3z_1 z_2 - z_2 z_3 + z_4$$

↓ make quantum

$$C(Z) = 3Z_1 Z_2 - Z_2 Z_3 + Z_4$$

→ Hamiltonian → variational algorithm

Quantum Approximate Optimization

search $|\vec{z}\rangle = |z_1, z_2, \ldots, z_n\rangle$ such that $C(Z)\,|\vec{z}\rangle = E_0\,|\vec{z}\rangle$

ansatz:

$$|\psi(\vec{\gamma}, \vec{\beta})\rangle = U_B(\beta_p)U_C(\gamma_p)U_B(\beta_{p-1})U_C(\gamma_{p-1})\ldots U_B(\beta_1)U_C(\gamma_1)\,|s\rangle$$

$$U_C(\gamma) = e^{-i\gamma C(Z)}$$

$$U_B(\beta) = e^{-i\beta \sum_j X_j}$$

$$|s\rangle = H^{\otimes n}\,|0, 0, \ldots, 0\rangle = \frac{1}{\sqrt{2^n}} \sum_z |z\rangle$$

variational algorithm

$$\rightarrow \quad \gamma_1, \gamma_2, \ldots, \gamma_p$$

$$\beta_1, \beta_2, \ldots, \beta_p$$

Quantum Approximate Optimization

search $|\vec{z}\rangle = |z_1, z_2, \ldots, z_n\rangle$ such that $C(Z)|\vec{z}\rangle = E_0|\vec{z}\rangle$

we know that solution is a product state

variational algorithm →

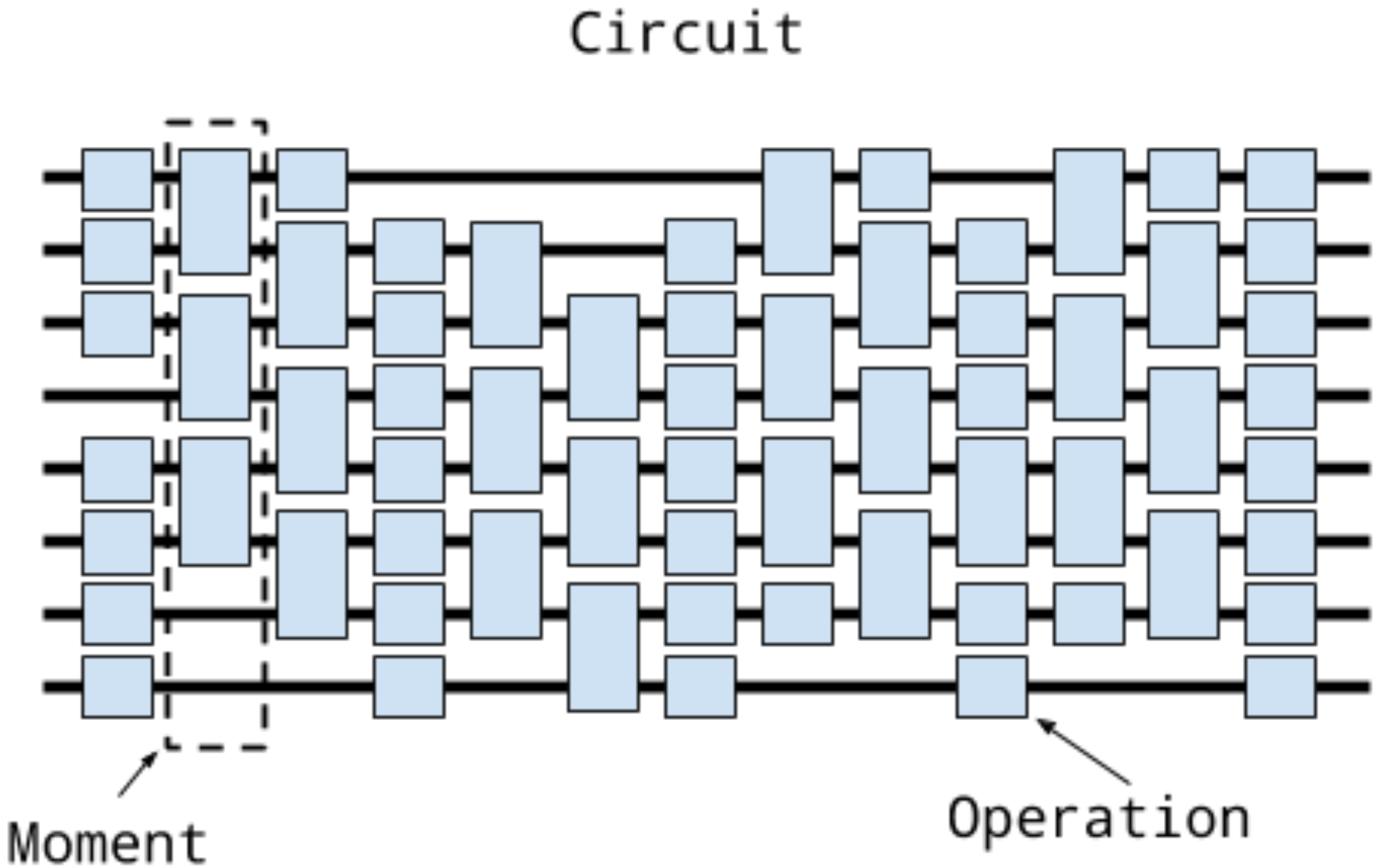
$$|\psi(\vec{\gamma},\vec{\beta})\rangle = \sum_{z_1,z_2,\ldots z_n=0}^{1} \alpha_{z_1,z_2,\ldots z_n} |z_1, z_2, \ldots z_n\rangle$$

will measure $z_1, z_2, \ldots, z_n$ with probability $|\alpha_{z_1,z_2,\ldots z_n}|^2$

for each measured $z_1, z_2, \ldots, z_n$ can compute $C(z)$

can find solution as long as it contributes to $|\psi(\vec{\gamma}, \overrightarrow{\beta})\rangle$

Cirq



Thanks!

And all the best for the exam!