Quantum Computing
Problem Set 7

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Problem 1: Phase Estimation

Consider a unitary U with an eigenvector $U|u\rangle = e^{i\phi}|u\rangle$ and $\phi = (2\pi/2^n)x$ with $x =$. We want to determine ϕ as accurately as possible, assuming that we can implement U and prepare one of it's eigenstates $|u\rangle$.

a) Investigate controlled-U operations $U|0\rangle|u\rangle = |0\rangle|u\rangle$ and $U|1\rangle|u\rangle = |1\rangle e^{i\phi}|u\rangle$. Describe a protocol where we apply U to $|+\rangle|u\rangle$, followed by a measurement of the control qubit, to infer information about ϕ. Which measurement can yield which information about ϕ? Will repeating the measurement improve the estimate?

b) Now consider a refined scheme where we assume we can also apply controlled-U^{2^k} operations for integer k efficiently. Starting with applying controlled-$U^{2^{n-1}}$, which information can one infer and what measurement should one make?

c) Now apply controlled-$U^{2^{n-2}}$ and take into account the result of the preceding step. What information can one infer and which measurement does we need to make? Rephrase the measurement as a unitary rotation followed by a measurement in the $|\pm\rangle$ basis.

d) Iterating the preceding steps, describe a procedure (circuit) to obtain ϕ exactly.

e) Compare the above procedure to the phase estimation algorithm discussed in the lectures.

Problem 2: Grover search

Grover's algorithm finds one marked item $\overline{x}$ in an unstructured list of $N = 2^n$ items. Classically you need to test on average $N/2$ items. Nonetheless, there is a quantum algorithm that can solve this problem with $\sqrt{N}$ queries. In the quantum setting there is the set of basis states $|x_j\rangle, j = 0, 1, 2, \ldots, N-1$ of the computational basis, i.e. each $|x_j\rangle = |00110\ldots\rangle$, and the goal is to find for which j one has $x_j = \overline{x}$.

a) Consider the operator U,

$$U|x\rangle = \begin{cases} -|x\rangle & \text{for} \quad x = \overline{x} \\ |x\rangle & \text{else} \end{cases} \tag{1}$$

which can also be written as

$$U = \mathbb{1} - 2|\overline{x}\rangle\langle\overline{x}| \tag{2}$$

Is there a way of constructing U in a similar manner as U_f used in Deutsch's problem?

b) Consider n qubits in states $|0\rangle$. Which state do you get after applying a Hadamard on each qubit and the U as in equation (1). Write the result in the basis of all states $|x_j\rangle, j = 0, 1, 2, \ldots N-1$. Hint, the amplitudes for all amplitudes of the basis states should be real.

c) Consider the action of the gate with operator

$$R = 2H^{\otimes n}|0,\ldots,0\rangle\langle 0,\ldots,0|H^{\otimes n} - \mathbb{1} \tag{3}$$

and evaluate it's action on a state

$$|\psi\rangle = \sum_j a_j |x_j\rangle \tag{4}$$

with real amplitudes a_j. How does R change the amplitudes a_j? Give a pictorial interpretation.

d) How does the probability of measuring the state $|\overline{x}\rangle$ change when applying R onto the state calculated in part **b)**? How does the probability of measuring the state $|\overline{x}\rangle$ change when applying R multiple times? How can one thus use the gates U and R to find the sought item $\overline{x}$?