Quantum Computing
Problem Set 2

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Problem 1: Time evolution of a qubit with dissipation

Consider a qubit that is, at time $t = 0$, in the pure state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \tag{1}$$

The coupling of the qubit to it's environment causes dissipation described be a master equation of the form

$$\dot{\rho} = -i[H,\rho] + \frac{\gamma_r}{2}(2\sigma^-\rho\sigma^+ - \sigma^+\sigma^-\rho - \rho\sigma^+\sigma^-) + \frac{\gamma_d}{2}(2\sigma^z\rho\sigma^z - 2\rho) \tag{2}$$

where $H = \omega\sigma^z$, the term with prefactor $\gamma_r > 0$ describes damping (including some dephasing) and the term with prefactor $\gamma_d > 0$ describes pure dephasing.

a) Write the density matrix $\rho(0)$ corresponding to the initial state $|\psi\rangle$ as given in equation (1).

b) Solve the master equation (2) for the case without Hamiltonian evolution $H = 0$.

c) Find the time evolution of $\langle\sigma^z\rangle$ and $\langle\sigma^x\rangle$ for the two cases of damping ($\gamma_r \neq 0, \gamma_d = 0$) and pure dephasing ($\gamma_r = 0, \gamma_d \neq 0$). How do the two dissipation processes affect the qubit?

Problem 2: No-cloning theorem

Consider a machine that could clone a quantum state, i.e. a machine that takes as input some system in an arbitrary state $|\phi\rangle$ together with a second system in a reference state $|0\rangle$, and generates the output $|\phi,\phi\rangle = |\phi\rangle|\phi\rangle$, that is it brings the second system into the state $|\phi\rangle$ while leaving the first system unperturbed. Assume that the machine could do this for any state $|\phi\rangle$.

a) Show that such a machine cannot exist as it would violate linearity. This is the no-cloning theorem, a fundamental restriction in the processing of quantum information and quantum states.

b) The no-cloning theorem does not forbid a transformation that would map

$$|0,0\rangle \to |0,0\rangle \quad \text{and} \quad |1,0\rangle \to |1,1\rangle \tag{3}$$

Onto which state does this transformation map the initial state $(|0\rangle + |1\rangle)|0\rangle/\sqrt{2}$? Write down a possible matrix for this transformation (there are several possibilities).

Problem 3: EPR pairs

Two parties, Alice and Bob are investigating the ability to communicate over long distances using quantum entanglement. They prepare a pair of qubits in the state $\alpha|00\rangle + \beta|11\rangle$. Alice takes the first qubit to the Moon. If Bob measures the second qubit on Earth (in the computational basis), the state will collapse to either $|00\rangle$ or $|11\rangle$, depending on the measurement outcome. Alice is then guaranteed to get the same outcome as Bob when she measures her qubit on the Moon.

a) For $\alpha = \beta = 1/\sqrt{2}$, show that the state $\alpha|00\rangle + \beta|11\rangle$ is entangled.

b) Suppose Alice knows that Bob made a measurement, but doesn't know the outcome. Conditioned on the outcome of Bobs measurement and thus on the post-measurement state what is the probability that Alice sees 0 or 1 when she subsequently measures her qubit.

c) Now suppose Bob never measured his qubit at all? Calculate Alices measurement probabilities for this case. Can Alice tell whether Bob made the measurement?